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ON THE SPECTRUM OF ANOSOV DIFFEOMORPHISMS

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ABSTRACT

Any Anosov diffeomorphism f induces a linear operator in the Banach space of continuous vector fields, which spectrum is contained in two annuli both being disjoint with the unit circle. The main theorem states that, if these annuli are sufficiently "narrow", then f has a fixed point. It is also proved that, if f acts on an infranilmanifold (e.g., on a torus), then the spectrum annuli for the algebraic automorphism corresponding to f are always more "narrow" than for f.

Let f be an Anosov diffeomorphism of a smooth compact Riemannian manifold M^n. The spectrum S of the induced linear operator in the space of continuous vector fields $(f_* v)(x) = df_{f^{-1}x} v(f^{-1}x)$ is contained in the interiors of two rings with radii $0 < r_1 < r_2 < 1$ and $1 < R_2 < R_1 < \infty$ (see [6]). It was shown in [1] and [2] that if

$$1 + \frac{\ln R_2}{\ln R_1} > \frac{\ln r_1}{\ln r_2} \tag{1}$$

or

$$1 + \frac{\ln r_2}{\ln r_1} > \frac{\ln R_1}{\ln R_2} \tag{2}$$

then: (i) the set NW(f) of the nonwandering points of f is M^n, (ii) the universal covering manifold $\bar{M}^n$ is homeomorphic to R^n, (iii) the fundamental group $\pi_1(M^n)$ has polynomial growth.

The following two theorems are the main results of this paper.

THEOREM 1. *Each condition* (1) *and* (2) *implies the existence of a fixed point for* f.

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THEOREM 2. *Let M^n be an infranilmanifold* (see [7], [8]), *and let f' be the infranilmanifold automorphism topologically conjugate to f (see* [5]). *Then one can choose the corresponding radii r'_i and R'_i for f' in such a way that $r'_1 \geqq r_1$, $r'_2 \leqq r_2$, $R'_2 \geqq R_2$, $R'_1 \leqq R_1$.*

Thus the spectrum of the linear model for f is not "wider" than the spectrum of f.

PROOF OF THEOREM 1. Suppose (1) is true. Let E^s, E^u be the stable and unstable subbundles of the tangent bundle TM^n and let W^s, W^u be the stable and unstable foliations of f. There exists a constant $C > 0$ such that for every positive integer k

$$C^{-1} r_1^k \| v \| < \| df^k v \| < C r_2^k \| v \| \qquad \text{for } v \in E^s, \tag{3}$$

$$C^{-1} R_2^k \| v \| < \| df^k v \| < C R_1^k \| v \| \qquad \text{for } v \in E^u. \tag{4}$$

Suppose the Riemannian metric is adjusted to f (see [4]), i.e., $C = 1$.

Let $\bar{f}$ be a covering diffeomorphism and let $\bar{W}^s$, $\bar{W}^u$ be the stable and unstable foliations of $\bar{f}$. It follows from [1] that for every two points $x, y \in \bar{M}^n$ the intersection $\bar{W}^s(x) \cap \bar{W}^u(y)$ consists of a single point $z = [x, y]$. Let us denote by $B_s(x, r)$ and $B_u(x, r)$ the r-balls on $\bar{W}^s(x)$ and $\bar{W}^u(x)$ respectively with the center x and by d_s and d_u the distances on the stable and unstable leaves respectively. For every $d > 0$ and $\delta > 0$ denote by $R(\delta, d)$ the supremum of those $\bar{R}$ for which $d_s([y, x_2], y) < d$ whenever $y \in B_u(x_1, \bar{R})$ and $x_2 \in B_s(x_1, \delta)$. It follows from [1] (see Lemma 3) that if d is small enough then there is a constant C_1 (independent of δ) such that

$$R(\delta, d) > C_1 \exp\left(\ln(d/\delta) \frac{\ln R_2}{\ln r_2 - \ln r_1}\right). \tag{5}$$

Fix a small $a > 0$ and choose $d > 0$ such that

$$\{[z, x_2] \mid z \in B_u(x_1, R(\delta, d))\} \subset B_u(x_2, (1 + a) R(\delta, d)) \tag{6}$$

whenever $d_s(x_1, x_2) < \delta$.

Now let $x \in \bar{M}^n$, $x_1 = \bar{f}^k x$, $x_2 = \bar{f}^k [x, fx]$. Inequality (5) implies that $d_s([z, x_2], z) < \delta$ if $z \in B_u(\bar{f}^k x, R(d_s(x_1, x_2), d))$. Let us denote $R(k) = R(d_s(x_1, x_2), d)$ and define the mapping $g : B_u(\bar{f}^k x, R(k)) \to \bar{W}^u(\bar{f}^k x)$ by $g(y) = \bar{f}^{-1}[y, \bar{f} y]$. It follows from (3) that

$$R(k) > C_2 \exp\left(k \frac{\ln R_2 \cdot \ln r_2}{\ln r_1 - \ln r_2}\right) \qquad \text{for some } C_2 > 0.$$

Thus (1) implies that $R(k) > C_2 \exp(k \ln(R_1 + \varepsilon))$ for some $\varepsilon > 0$. On the other hand

$$d_u(\bar{f}^{k+1}x, [\bar{f}^k x, \bar{f}^{k+1}x]) < \exp(k \cdot \ln R_1) \cdot d_u(x, [x, \bar{f}x])$$

and

$$\bar{f}^{-1}B_u(\bar{f}^{k+1}x, (1+a)R(k)) \subset B_u(\bar{f}^k x, (1+a)R_1^{-1}R(k))$$

(see (4)). Therefore, if $1 + a < R_1$ and k is large enough then g maps the ball $B_u(\bar{f}^k x, R(k))$ into itself, hence g has a fixed point. Let $g(y_0) = y_0$, i.e. $\overline{f\bar{W}^s(y_0)} = \bar{W}^s(y_0)$. Since $\bar{f}$ is a contraction on $\bar{W}^s(y_0)$ it also has a fixed point the projection of which to M^n is a fixed point of f. Q.E.D.

Proof of Theorem 2. It is sufficient to show that the statement of the theorem is true if M^n is a nilmanifold (otherwise one can consider a coverging diffeomorphism of the covering nilmanifold). So let $M^n = N/G$ where N is a nilpotent simply connected Lie group and G is a uniform discrete subgroup. According to A. Manning f is topologically conjugate to a nilmanifold automorphism A. There exists a covering transformation $\bar{f}$ such that $\bar{f}(e) = e$, where e is the identity of N. Note that the set of diffeomorphisms of M^n is dense in the set of homeomorphisms and that every two diffeomorphisms which are diffeomorphically conjugate have the same spectrum. Hence without loss of generality one can suppose that f is C^0-close to A. So f and A induce one and the same automorphism of G and the homeomorphism h for which $fh = hA$ is C^0-close to the identity mapping and induces the identical automorphism of the fundamental group G. Let $\bar{h}$ be the covering mapping such that $\bar{h}(e) = e$ and $\bar{A}$ be the covering automorphism of the group N. If s belongs to the spectrum of $\bar{A}$ then s is an eigenvalue and there is a one-parameter subgroup g_t such that $d(\bar{A}g_t, e) = |s| d(g_t, e)$. Suppose now that $|s| > 1$. The distance $d(\bar{A}^k g_t, e)$ increases like $|s|^k$ and so does the distance $d(\bar{h}\bar{A}^k g_t, e)$ because h is homotopic to the identity. Thus the distance $d(\bar{f}^k \bar{h} g_t, e)$ also increases like $|s|^k$. Note that $\bar{f}^k \bar{h} g_t \in \bar{W}^u_{\bar{f}}(e)$ and that the foliation $\bar{W}^u_{\bar{f}}$ is the lifting of W^u_f. Dividing the curve $\bar{A}^k g_t$ into pieces and considering their images under $\bar{h}$ one can easily prove that the distance between $\bar{f}^k \bar{h} g_t$ and e measured along the leaf $\bar{W}^u_{\bar{f}}(e)$ also increases like $|s|^k$. This fact implies that $R_2 \leqq |s| \leqq R_1$. Hence $R_2' \geqq R_2$. The inequality $R_1' \leqq R_1$ is treated by analogy. To prove the other two inequalities it is sufficient to consider $\bar{f}^{-1}$ and $\bar{A}^{-1}$. Theorem 2 is proved.

Corollary 3. *If f is an Anosov diffeomorphism of an infranilmanifold M^n and both* (1) *and* (2) *are true then M^n is a torus.*

This statement follows from [1] (see Proposition 7) and from Theorem 2.

Remark 4. In [1] I conjectured that, if (1) or (2) holds, then M is a torus.

D. Fried constructed a counterexample to this conjecture (see [3]). The counterexample is an Anosov automorphism of a nilmanifold. Corollary 3 shows, however, that if both (1) and (2) are satisfied, then M is a torus (provided M is a nilmanifold).

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